**Chapter 3**

**INTEGRAL, DIFFERENTIAL, AND JUMP EQUATIONS DERIVED FROM CONSERVATION LAWS**

In continuum mechanics, analysis of various systems leads to models where density, velocity and stress distributions are discontinuous integrable functions. The mass, linear momentum, angular momentum and energy equations used in such models should be derived from integral equations.

**§1. Three Theorems about Volume and Boundary Surface Integrals of Differentiable Functions**

Consider three important theorems about integrals.

**The time derivative of a volume integral.** Consider the time derivative of an integral over a moving (time-dependent) domain:

 (**x**, *t*) d*V* =  =

=  *t* +Δ*t*) d*V* – + 

–  (**x**, *t*)  = (3.1.1)

=  *t* +Δ*t*) – *f*(**x**, *t*)] d*V* + .

Fig. 3.1.1. Time-dependent domain *V* bounded by moving surface *S* at times *t* and d*t*.

**n**(*t*)

M′

M

*S*(*t* + Δ*t*)

δ*n* < 0

**n**(*t*+Δ*t*)

M

M′

*S*(*t*)

δ*n* > 0

Figure 3.1.1 shows schematically domains *V*(*t*) and *V*(*t* +Δ*t*), their boundary surfaces *S*(*t*) and *S*(*t* +Δ*t*) (drawn in bold and thin lines, respectively), and outward (relative to *V*(*t*)) unit normal vectors **n** to *S*(*t*) and *S*(*t* +Δ*t*). The domain *V*(*t* +Δ*t*) – *V*(*t*) can be represented as a union of cylinders of volume

δ*V* = (Δ**x** ⋅ **n**) d*S* (Δ**x** ⋅ **n** ≡ δ*n*), (3.1.2)

where the displacement Δ**x** of a surface point over time interval Δ*t* establishes a pairwise correspondence between points on *S*(*t*) at the time-interval endpoints. Only point-to-point displacement, or correspondence, along the normal **n** to *S*(*t*) is essential for further analysis. The sign of Δ**x** ⋅ **n** ≡ δ*n* determines whether the contribution of a volume element δ*V* to the integral over *V*(*t* +Δ*t*) – *V*(*t*) in (3.1.1) is positive or negative. If the local displacement of *S*(*t*) is away from *V*(*t*), then δ*n* > 0 and δ*V* > 0; if it is into *V*(*t*), then δ*n* < 0 and δ*V* < 0.

Given the time dependence of the boundary surface *S*(*t*), or the point-to-point correspondence, as a function Δ**x** = Δ**x**(**x**, *t*), one can find the rate of surface displacement along the outward normal at every point of *S*(*t*):

*N* = . (3.1.3)

Here, *N* > 0 if the local displacement of *S*(*t*) is in the direction of **n** (away from *V*(*t*)), and *N* < 0 if the displacement of *S*(*t*) is opposite to **n** (into *V*(*t*)). Examples of definition and calculation of the normal displacement velocity *N* of *S*(*t*) are given below.

Returning to Eq. (3.1.1), we obtain

 (**x**, *t*) d*V* =  d*V* +

+ *t*) . (3.1.4)

In summary, we have the following:

**Theorem 1** (***time derivative of an integral over a moving volume***). *The time derivative of a volume**integral over the domain V*(*t*) *bounded by a surface S*(*t*)*, where the integrand f*(***x****, t*) *is a differentiable function of time t and the surface S*(*t*) *has a local normal velocity N, is given by the expression*

(**x**, *t*) d*V* =  d*V* +  (**x**, *t*) *N*(**x**, *t*) d*s*. (3.1.5)

Consider two types of control volume in a continuous medium.

One of these is the Eulerian control volume *V*E bounded by a surface *S*E (see also Chapter 1, §4), where *V*E and *S*E are time-independent in the observer's coordinate system; i.e., the boundary surface *S*E is stationary and *N* = 0:

*V*E = const, *S*E = const, *N* = 0. (3.1.6)

The other is the Lagrangian (material) control volume *V*L bounded by a surface *S*L (see also Chapter 1, §3), where *V*L and *S*L move with the medium; i.e., *VL* consists of the same particles. Both *V*L(*t*) and *S*L(*t*) can be defined in this manner by virtue of the axiom of particle identity (see Chapter 1, §1).

Both *V*E and its boundary surface *S*E are time-independent, whereas changes in *V*L and its boundary surface *S*L are determined by the local velocity field:

*V*L= *V*L(*t*), *S*L = *S*L(*t*),  = **v**(**x**, *t*), *N* = **v** ⋅ **n** ≡ *vn*. (3.1.7)

For a stationary (Eulerian) control volume, *N* = 0 at every point of *S*E, and it follows from (3.1.5) that the volume integration and time differentiation operators are commutative:

 (**x**, *t*) d*V* = d*V*; (3.1.8)

i.e., the time derivative of an integral of a function over an Eulerian control volume is equal to the corresponding volume integral of the time derivative of the function.

For a Lagrangian control volume, *N* = 0 at every point of *S*E, and it follows from (3.1.5) that

(**x**, *t*) d*V* = d*V* +(**x**, *t*) *vn*(**x**, *t*) d*s*; (3.1.9)

i.e., time differentiation and integration over a Lagrangian control volume are not commutative.

As an example, consider the calculation of the normal velocity *N* for the surface *S*(*t*) defined by the equation

ψ(*x*1, *x*2, *x*3, *t*) = 0. (3.1.10)

By choosing a function ψ such that its values at points outside *V*(*t*) are greater than those at points lying on *S*(*t*) at time *t*, the outward unit normal vector can be determined at every point of *S* at any time as

**n** =  (3.1.11)

and used in the expansion of ψ(**x**, *t*),

ψ(**x**+Δ**x**, *t* +Δ*t*) – ψ(**x**, *t*) = Δ*xi* + Δ*t* + o(Δ*x*, Δ*t*). (3.1.12)

Rewritten componentwise, this expansion reads

ψ(*x*1 + Δ*x*1, *x*2 + Δ*x*2, *x*3 + Δ*x*3, *t* +Δ*t*) – ψ(*x*1, *x*2, *x*3, *t*) = (3.1.12a)

= Δ*x*1 + Δ*x*2 + Δ*x*3 + Δ*t* + o(Δ*x*, Δ*t*).

Let М be a point with coordinates (**x**, *t*) = (*x*1, *x*2, *x*3, *t*) that lies on the surface *S*(*t*) at time *t* (see Fig. 3.1.1). At time *t* +Δ*t*, point М′ with coordinates (**x** + **n** δ*n*, *t* +Δ*t*) ≡ (*x*1 + *n*1δ*n*, *x*1 + *n*2δ*n*, *x*3 + *n*3δ*n*, *t* +Δ*t*) lies on the surface *S*(*t* +Δ*t*). Therefore,

ψ(**x**, *t*) ≡ ψ(*x*1, *x*2, *x*3, *t*) = 0, (3.1.13)

ψ(**x** + **n** δ*n*, *t* +Δ*t*) ≡ ψ(*x*1 + *n*1δ*n*, *x*1 + *n*2δ*n*, *x*3 + *n*3δ*n*, *t* +Δ*t*) = 0.

If Δ**x** = **n** δ*n* (Δ*xi* = *ni* δ*n*), then it follows from (3.1.12) and (3.1.13) that

*ni* δ*n* + Δ*t* + o(δ*n*, Δ*t*) = 0. (3.1.14)

By using (3.1.11), the first term (with contraction over *i*) is represented as

δ*n* *ni* = δ*n* (****ψ ⋅ **n**) = δ*n * = δ*n * = δ*n* |****ψ|, (3.1.15)

and (3.1.14) becomes

δ*n* |****ψ| + Δ*t* + o(δ*n*, Δ*t*) = 0. (3.1.16)

Recalling the definition of *N* in (3.1.3), we obtain

*N* =  = – . (3.1.17)

**Transformation of a surface integral into the volume integral of a divergence: The Gauss–Ostrogradsky theorem.** The rate of change of a material volume element δV is determined by the normal velocity at its boundary δS:

(δ*V*) =  = *ni* d*s*. (3.1.18)

However, according to (1.8.24),

(δ*V*) = *eii* = ∇*ivi*. (3.1.19)

Therefore,

*ni* d*s* = δ*V* ⋅ ∇*ivi* + o(δ*V*), (3.1.20)

where ∇*ivi* is taken at the center of δ*V*. Any finite volume *V* can be represented as a union of volume elements δ*V*(*k*) small enough that the change in ∇*ivi* across δ*V*(*k*) is negligible ( << *L*, where *L* is a characteristic length of the problem). Summing up Eqs. (3.1.20) over all volume elements yields

 =  + . (3.1.21)

The integrals over the volume-element boundaries δ*S*(*k*) lying inside *V* cancel out since the respective outward normals to the interface between each pair of contiguous volume elements are the negatives of one another at each point of the interface:

 = –;  = –.

Therefore, it holds for any partition that

*ni* d*s* = *ni* d*s*. (3.1.22)

In the limit of *K* → ∞ (max(δ*V*(*k*)) → 0), the former sum on the right-hand side of (3.1.21) becomes a volume integral over *V* while the latter vanishes:

∇*i vi*)*k* δ*V*(*k*) → ∇*i vi*) d*V*,

 = *K* ⋅ o(δ*V*(*k*)) =  *o*(δ*V*(*k*)) → 0. (3.1.23)

As a result, we have

*n i* d*s* = *vi* d*V*. (3.1.24)

Any vector field **A** = *Ai* **e***i* can be viewed as a velocity field, and an equation similar to (3.1.24) is valid, with *vi* standing for *Ai*. It can easily be shown that an analogous result holds for a tensor field of arbitrary rank. Thus, the following theorem is true:

**Theorem 2** (***Gauss–Ostrogradsky­***). For any differentiable tensor field of arbitrary rank defined on an arbitrary domain V, bounded by a surface S, the surface integral of the normal component *An = Ai ni* is equal to the volume integral of the divergence div ***A*** *=* ∇*i Ai*:

*ni* d*s* =  d*V*. (3.1.25)

For a two-dimensional vector field

*Bi* = *Bi*(*x*, *y*), *i* = *x*, *y*,

whose domain  is bounded by a closed curve *L*, the analog of Gauss–Ostrogradsky­ theorem (3.1.25) states that the line integral of *Bi* along *L* is equal to the integral of its divergence over :

 = d*s*. (3.1.25a)

This equation follows from Eq. (3.1.25) written for a field *Bi*(*x*, *y*) defined on a domain *V* whose boundary consists of a cylindrical surface *S*cyl with generator parallel to the *z* axis, a "bottom" base (at *z* = 0) coinciding with , and a "top" base  (at *z* = *Н*) parallel to the *x*, *y* plane. The integrals of *Bini* over and  cancel out because the corresponding outward normal vectors are antiparallel. Therefore,

 d*s* = +d*s*

*= Н* ,

 = d*V* = *Н* d*s*,

and Eq. (3.1.25a) follows immediately.

Applying Gauss–Ostrogradsky theorem (3.1.25) to the surface integral over *S*L in expression (3.1.9) for the derivative of an integral over a time-dependent Lagrangian control volume *V*L, we obtain

(**x**, *t*) d*V* = d*V*. (3.1.26)

**Zero integrand theorem.** The following is a sufficient condition that the integrand is identically zero:

**Theorem 3** (zero integrand theorem). If a continuous function Φ(**x**, *t*) defined on a domain *V* is such that, for any subdomain Δ*V* in *V*,

 d*V* = 0, (3.1.27)

then

Φ = 0 (3.1.27a)

throughout the domain *V*.

*Proof by contradiction.* Suppose that *V* contains a point М where Φ ≠ 0 (say, Φ > 0). By continuity of Φ(**x**), there exists a neighborhood Δ*V* containing *М* where Φ > 0. Then,

 d*V* > 0;

i.e., *V* contains a subdomain Δ*V* where condition (3.1.27) is violated. Since this condition must hold for any subdomain, Eq. (3.1.27a) holds at every point in *V*.

**§2. Integral and Differential Mass Balance Equations**

Consider the Eulerian control volume *V*E bounded by a surface *S*E (time-independent in the observer's coordinate system) across which mass is transferred. An Eulerian description is employed; i.e., the distributions of density, velocity, stress, body forces, and other variables are functions of (*x*1, *x*2, *x*3, *t)*. In accordance with basic principles of continuum mechanics, these functions are assumed to be continuous almost everywhere (with the possible exception of singular surfaces, curves, and points). Therefore, they can be integrated over a volume or surface.

To describe the mass balance within an Eulerian control volume, the mass *mV*(*t*) therein and its rate of change  are represented as a volume integral and its time derivative, respectively:

*mV*(*t*) = (**x**, *t*) d*V*,

 =  d*V.* (3.2.1)

**Theorem (*mass conservation*).** The mass within a time-independent domain can only be changed by mass *transfer* across its boundary.

According to (1.4.14), the rate of mass transfer is the surface integral of mass flux **j** = ρ**v** over *S*E. Thus, *the mass balance for an Eulerian control volume* *V*E *has the form of an integral equation:*

 d*V* = –*vk nk* d*S*, (3.2.2)

where **n** = *nk* **e***k* is the outward unit normal to *S*E.

When the functions ρ(**x**, *t*) and **v**(**x**, *t*) are differentiable with respect to time and coordinates, the time derivative of the volume integral over *V*E in Eq. (3.2.2) can be reduced to (3.1.8) by applying Theorem 1 (see § 1),

 d*V* = d*V*, (3.2.3)

and the surface integral over *S*E can be expressed by applying the Gauss–Ostrogradsky­ theorem (Theorem 2, or (3.1.25)):

–*vk nk* d*s* = –(ρ*vk*) d*V*. (3.2.4)

Then, mass balance equation (3.2.2) is rewritten as

 d*V* = 0. (3.2.5)

Since this integral is zero for any control volume *V*E, it follows from Theorem 3 that (see §1, Eqs. (3.1.27) and (3.1.27a)) that the integrand is identically zero at each point:

 + ∇*k*(ρ*vk*) = 0. (3.2.6)

This differential equation of mass conservation, known as the continuity equation, is derived above by a different method (see (1.9.14)).

Analogously, for the Lagrangian control volume *V*L bounded by a surface *S*L (i.e., a body consisting of the same particles), mass conservation implies that the mass *mV* therein does not change:

*mV* = d*V*,  = 0.

Thus, *the mass balance law for a Lagrangian control volume V*L *has the form*

d*V* = 0. (3.2.7)

By analogy with (3.2.3), when the functions ρ(**x**, *t*) and **v**(**x**, *t*) are differentiable with respect to time and coordinates, the time derivative of the volume integral over a time-dependent (Lagrangian) control volume can be reduced to (3.1.26) by applying Theorem 1 and the Gauss–Ostrogradsky­ theorem (Theorem 2):

d*V* = d*V* = 0. (3.2.8)

Since this equation holds for any control volume *V*L, differential equation (3.2.6) of mass conservation follows immediately.

Note that expression (3.1.26) for the derivative of the integral of a function ρ*f* over a Lagrangian control volume *V*L (where *f* is an arbitrary continuously differentiable function) can be rewritten as

*f* d*V* = d*V* =  d*V* , (3.2.9)

where use is made of Eq. (3.2.4).

Combined with continuity equation (3.2.6), this leads to the following equations to be used in the analysis below:

. (3.2.10)

**§3. Integral and Differential Momentum Equations**

The momentum **J***V* in an Eulerian control volume *V*E and its rate of change d**J***V /*d*t* are defined as an integral and a corresponding derivative, respectively:

**J***V* = **v** d*V*,  = **v** d*V*. (3.3.1)

Linear momentum conservation implies that the momentum within an Eulerian control volume can be changed primarily by the momentum transfer due to mass transfer across its boundary. (Recall that momentum, energy, entropy, enthalpy, and other extensive properties are carried along with incoming or outgoing matter.) This mechanism is known as convective momentum transfer. The rate of momentum transfer, given by expression (1.4.15) with *f* = **v**, is

**v** d*S*. (3.3.2)

However, the momentum within *V*E can also be changed by external forces, which include the contact forces[[1]](#footnote-1) acting on *S*E and the body forces within *V*E:

d*S* + d*V* (**F** = **g′** + **R**, **g′** = **g** +), (3.3.3)

where the body force per unit mass **F** is the sum of a gravitational contribution (determined by gravitational acceleration **g**), the inertial force density **** in a non-inertial frame of reference, and the ponderomotive force density **R**. Gravitational and inertial forces act as a unified force, which is generally represented by an external acceleration **g′**.

In summary, since *jn* = ρ*vk nk* and = *nk*, we obtain an integral representation of *the linear momentum balance for an Eulerian control volume* *VE*:

**v** d*V* = –*vk nk* **v** d*s* + d*s* + **F** d*V*. (3.3.4)

This momentum balance equation can be expressed in words as follows:

**Theorem (*momentum conservation*).** The momentum within a time-independent domain can be changed by

- the momentum transfer due to mass transfer across the domain boundary;

- external forces on the boundary;

- body (gravitational, electromagnetic, inertial) forces within the domain.

An analogous momentum equation is obtained for the (time-dependent) Lagrangian control volume *V*L bounded by a surface *S*L (i.e., a body consisting of the same particles). Since the momentum of the body can only be changed by external forces (3.3.3), *the linear momentum balance for a Lagrangian control volume* *V*L can be expressed by the integral equation

**v** d*V* = *nk* d*s* + **F** d*V*. (3.3.5)

In this equation, the term representing convective transfer across the boundary is included in the time derivative of the integral over a Lagrangian control volume. In view of (3.1.9), it is clear that Eq. (3.3.5) is equivalent to (3.3.4).

When the density ρ, velocity **v**, and stress σ*ik* distributions are differentiable, one can use (3.1.8) to express the time derivative of an integral over an Eulerian control volume and transform the surface integral into a volume integral by applying Gauss–Ostrogradsky­ theorem (3.1.25), as in (3.2.3)–(3.2.5), to obtain

d*V* = 0. (3.3.6)

Since the Eulerian control volume *V*E considered here is arbitrary, Theorem 3 implies that the following equation is valid at every point where the functions in the integrand are differentiable:

ρ **v** + ∇*k* (ρ **v** *vk*) = ∇*k* **σ***k* + ρ **F**. (3.3.7)

By expanding Einstein's summation, this equation is rewritten as

(ρ **v**) + (ρ **v** *v*1) + (ρ **v** *v*2) + (ρ **v** *v*3)

=  +  +  + ρ **F**. (3.3.8)

Representation (3.2.8) and Eq. (3.2.10) with *f* ≡ **v** are then used to transform momentum equation (3.3.7) into the Cauchy equation of motion:

ρ =  + ρ **F** (3.3.9)

.

This equation can be interpreted as Newton's second law for a continuum particle written in terms of intensive variables: mass (ρ) times acceleration (d**v** / d*t*) equals the net force on the particle, which includes surface and body forces (∇*k* **σ***k* and ρ**F**).

It is obvious that

**v** = + *vk* ∇*k* ≡ . (3.3.10)

Hence, taking the dot product of Eq. (3.3.9) with **v**, we obtain the kinetic energy equation for a continuum particle, sometimes referred to as *vis vita equation*:

ρ= **v** ⋅ ∇*k***** + ρ **F** ⋅ **v** ≡ *vl* ∇*k* **σ***lk* + ρ *Fl* *vl*. (3.3.11)

**§4. Integral and Differential Angular Momentum Equations**

Following §4 in Chapter 2, let us consider first the case when the angular momentum **(**moment of momentum) of a continuum is completely determined by the velocity field **v**(**x**, *t*). This means that the angular momentum **K***V* about the origin О (base of the position vector **x**) within an arbitrary control volume *V* is

**K***V* =**x** × **v**] ρ d*V*. (3.4.1)

Furthermore, assume that the moment of external forces is entirely due to surface forces on the control volume boundary *S* (represented in terms of stress σ*kl*) and body forces (represented by a force density vector field ρ**F**):

**x** × ] d*S* + **x** × ρ**F**] d*V*. (3.4.2)

Note that expressions (3.4.1) and (3.4.2) should be generalized in order to describe micropolar media, in particular by taking account of an intrinsic spin unrelated to the macroscopic velocity field **v**(**x**, *t*) (see discussion at the end of this section). However, the present analysis is restricted to the representation of angular momentum based on (3.4.1) and (3.4.2) unless specified otherwise.

Angular momentum conservation implies that the angular momentum within an Eulerian control volume *V*E (time-independent in the observer's coordinate system) can only be changed by

- the convective angular momentum transfer due to mass transfer across its boundary *S*E (see (1.4.12));

- the torque due to external forces on *S*E;

- the torque due to body forces within *V*E.

In summary, *the* *angular momentum balance for an Eulerian control volume* is expressed by the integral equation

**x** × **v**] ρ d*V*

= −*vn* [**x** × **v**] d*S* + **x** × ] d*S* + **x** × ρ**F**] d*V*. (3.4.3)

This angular momentum balance equation, analogous to mass balance (3.2.2) and momentum balance (3.3.5), can be expressed in words as follows:

**Theorem (*angular momentum conservation***). The angular momentum within a time-independent domain can be changed by

**-** the angular momentum transfer due to mass transfer across the domain boundary;

**-** the torque due to external forces on the boundary;

**-** the torque due to body (gravitational, electromagnetic, inertial) forces within the domain.

An analogous angular momentum equation is obtained for the (time-dependent) Lagrangian control volume *V*L bounded by a surface *S*L (i.e., a body consisting of the same particles). The angular momentum of the body about the origin can only be changed by the torque due to external forces (3.3.3). As a result, *the* *angular momentum balance for a Lagrangian control volume* *V*L is expressed by an integral equation that contains no terms representing convective transfer across the boundary:

**x** × **v**] ρ d*V* = **x** × ] d*s* + **x** × ρ**F**] d*V*. (3.4.4)

When the density ρ, velocity **v**, and stress σ*ik* distributions are differentiable, one can use (3.1.8) to express the time derivative of an integral over an Eulerian control volume and transform the surface integral into a volume integral by applying Gauss–Ostrogradsky­ theorem (3.1.25), as in (3.2.3)–(3.2.5) and (3.3.6). Finally, Theorem 3 (about zero integrand) is applied to show that local balance of angular momentum is described by a differential equation analogous to (3.2.6) and (3.3.7):

([**x** × **v**] ρ) + ∇*k*(ρ *vk* [**x** × **v**]) = ∇*k*[**x** × ] + ρ [**x** × **F**]. (3.4.5)

As with mass or linear momentum balance, this differential equation can be derived by considering an arbitrary material volume *V*L.

By the product rules for the operators ∂ /∂*t* and ∇*k*, it holds that

[**x** × **v**] ρ = [**x** ×  ρ **v**], (3.4.6)

∇*k*([**x** × **v**] ρ *vk*) = [(∇*k* **x**) × **v**] ρ *vk* + [**x** × ∇*k* (**v** ρ *vk*)].

Furthermore,

∇*k* **x** = ∇*k* (*xi* **e***i*) = **e***i* ∇*k* *xi* = **e***i* δ*ki* = **e***k*,

∇*k*[**x** × **σ***k*] = [∇*k* **x** × **σ***k*] + [**x** × ∇*k***σ***k*]. (3.4.7)

Then, the first terms in the latter identities above are transformed by using (A1.9.9) as

[(∇*k* **x**) × **v**] ρ*vk* = [**e***k* × **v**] ρ*vk* = [*vk* **e***k* × **v**] ρ = [**v** × **v**] ρ = 0, (3.4.8)

[(∇*k* **x**) × **σ***k*] = [**e***k* × **σ***k*] = **e***i* ε*ilm* **e***kl* σ*mk* = **e***i* ε*ilm* δ*kl* σ*mk* = **e***i* ε*ikm* σ*km*,

and Eq. (3.4.5) becomes

[**x** × (ρ **v**)] + [**x** × ∇*k*(ρ *vk* **v**)] = [**x** × ∇*k*] + [**x** × ρ **F**] + **e***i* ε*ikm* σ*km*.

Collecting the cross products, we obtain

 = **e***i* ε*ikm* σ*km*. (3.4.9)

Since the left-hand side of this equation is zero by virtue of momentum equation (3.3.7), the differential angular momentum equation based on representation (3.4.1), (3.4.2) reduces to the algebraic equation

ε*ikm* σ*mk* = 0. (3.4.10)

Rewritten componentwise, this equation implies the symmetry of the stress tensor, which is established above by analyzing the angular momentum balance for an infinitesimal material cube (see (2.4.9)):

*i* = 1: σ32 – σ23 = 0,

*i* = 2: σ13 – σ31 = 0, (3.4.11)

*i* = 3: σ21 – σ12 = 0.

Thus, the angular momentum balance based on representation (3.4.1), (3.4.2) implies the symmetry of the stress tensor only and does not lead to any additional constraints.

Consider the angular moment of a particle occupying a volume Δ*V* that is small in the sense that its size Δ*x* is much smaller than the length scale *L* of variation of ρ and **v**. Let М be the center of mass of the particle:

**x**(M) Δ*m* = ρ d*V*, Δ*m* = d*V*, (3.4.12)

or (equivalently)

ρ d*V* = 0 (δ**x** = **x** – **x**(*M*)). (3.4.13)

Hereinafter, superscript (М) refers to values at point М.

The contribution of the regular velocity distribution in the particle (which undergoes rotation and deformation) to the angular momentum within Δ*V* is determined as follows. The velocity distribution is represented as

**v** = **v**(M) + (∇*k* **v**)(M) δ*xk* + *v*0 O** (3.4.14)

(see Eq. (1.8.8) and expansion below it).

According to (3.4.1), the angular momentum within Δ*V* is

Δ**K** =  d*V* = (3.4.15)

.

Note that the quantities with subscript (М), whose values are taken at point М, are fixed and can therefore be factored out of the integral:

Δ**K** = [**x**(M)× **v**(M) ] d*V*

+  + 

+  + *x*(M) *v*0 O ** d*V*.

Here, the second and third integrals vanish by virtue of (3.4.13) since δ*xk* is measured from the center of mass, and the estimate (∇*k vl*)(M) ~ *v*0 /*L* (cf. (1.8.16)) is used to represent the fourth and fifth integrals as

*Lv* *v*0 O ** Δ*m* + *x*(M) *v*0O** Δ*m*.

In summary, the angular momentum per unit mass of a small particle is represented as

 = [**x**(M) × **v**(M)] + *Lv* *v*0 O ** + *x*(M) *v*0 O **. (3.4.16)

Thus, the angular momentum of a particle of size small enough that Δ*x* << *L*, with a regular velocity distribution therein, is determined by its center-of-mass motion (recall Cauchy–Helmholtz theorem (1.8.15)), whereas the contribution of motion relative to the center of mass (rotation, deformation) is smaller by two orders of magnitude with respect to the relative particle size (O [(Δ*x*/*L*)2]).

Consequently, representation (3.4.1) is tantamount to defining local specific angular momentum of a particle (per unit mass) as the moment of its velocity relative to the origin:

**k** =  = [**x**(M) × **v**(M)]. (3.4.17)

**Intrinsic angular momentum (spin), surface and body couples (torques).** In some materials of practical importance (micropolar media), angular momentum contains a significant contribution from micro-rotations (intrinsic spin), such as electron and nuclear spins, electron orbital motion, or intrinsic rotations of particles in a disperse system (Fig. 3.4.1). Micro-motions of this kind cannot be described in terms of **v**(**x**, *t*) because they do not contribute to the macroscopic velocity field.

Рис. 3.4.1. Microscopic particles with intrinsic spin.

М



It is clear from Fig. 3.4.1 that the angular momentum of a volume element should be evaluated by taking into account not only its rotation as a whole (e.g., about its center of mass at point М) but also the spin density **m** of rotating particles:

Δ**K** = [**x**(М) × **v**] ρ Δ*V* + ρ **m** Δ*V*, ρ **m** = *n* *I*. (3.4.18)

Here, the microscopic particles represented by hatched circles are identical and rotate with the same angular velocity  kinematically independent of the macroscopic velocity field **v**(**x**, *t*), *n* is the number of rotating particles per unit volume, and *I* is the moment of inertia of a particle relative to its center of mass.

In some ferromagnets, the spin velocity ω2 and number density *n* of aligned rotating particles are very high. In such materials, even if *I* is small (as when the particles are atoms), the contribution due to  = *nI*ω2 can many times exceed those of the second and third terms in (3.4.16), which are determined by the regular velocity field **v**(**x**, *t*). Moreover, it can even be comparable to the leading-order (first) term. One should keep in mind here that the intrinsic angular momentum represented by **m** can be significant only if two conditions are satisfied. First, the micro-rotations must have a preferred orientation (as do particle spins in Fig. 3.4.1); otherwise, their angular momenta cancel out. Second, the spin velocity ω2 must be much higher than the magnitude ω =  of the angular velocity associated with the **v**(**x**, *t*) field:

ω2 >> ∇*k* *vj*, ω2 >> ω = |rot **v**|. (3.4.19)

Otherwise (when ω2 ~ ω), the angular momentum due to **m** is comparable to that of macroscopic rotation about the particle's center of mass but is not sufficiently higher than the second and third terms in (3.4.16), whose contributions to local angular momentum balance are negligible. Then, the stress tensor is symmetric, as expressed by Eq. (3.4.10).

As a manifestation of intrinsic angular momentum, consider the following simple experiment illustrating the so-called gyromagnetic effect, also known as the Einstein–de Haas effect. An initially unmagnetized ferromagnetic rod is suspended vertically inside an electromagnetic coil. When a magnetic field of strength **H** is applied, the magnetic domains of the rod tend to align parallel to the field, and the magnetized rod acquires a nonzero intrinsic angular momentum with density **m**(**H**) even though its macroscopic velocity is zero. Once the source of the magnetic field is switched off, thermal motion destroys the alignment of magnetic domains, the spins become randomly oriented, and the resultant intrinsic angular momentum vanishes (**m** = 0). However, since no torque is applied to the rod, its total angular momentum

d*V*

must be equal to ρ**m**(**H**)*V*, where *V* is the volume of the rod and ρ = const. As a result, the rod acquires a macroscopic angular momentum and starts to rotate. Since the corresponding linear velocity is **v** = [**x** × **ω**], its angular velocity **ω** is determined by conservation of the total angular momentum between the unmagnetized (**m** = 0) and magnetized (**ω** = 0, **m**(**H**) ≠ 0) states:

d*V* = **m**(**H**) *V*.

The gyromagnetic effect described above demonstrates the significant contribution of intrinsic spin to angular momentum balance.

Intrinsic angular momentum, represented in terms of the corresponding field **m**(**x**, *t*), can be created and sustained by (distributed) body couples **M**(**x**, *t*) analogous to body forces (e.g., by an electromagnetic field) and surface (contact) couples **** = ****(**x**, *t*, **n**) analogous to surface (contact) forces. The latter depend not only on position and time but also on the normal **n** to the surface element on which they act (as is ****). Accordingly, angular momentum balance (3.4.3) is expressed in generalized form as

([**x** × **v**] + **m**) d*V* = –*vn* ([**x** × **v**] + **m**) d*S*

+  + d*V*. (3.4.20)

By applying this equation to Cauchy's tetrahedron (see Fig. 2.2.1), it can be shown that the surface couple vector **** can be represented, similarly to **** in (2.2.15), in terms of surface torques ****, ****, **** acting on the coordinate planes:

**** ≡ ****(**x**, *t*, **n**) = *****nk*, **** ≡ ****(**x**, *t*, ), ****= θ*lk* . (3.4.21)

Then, by following the derivation of (2.2.21), it is shown that the torque components θ*lk* are those of a second-rank *couple stress* tensor (denoted by **T**) in the coordinate system spanned by the unit vectors .

After transformations analogous to those leading from (3.4.3) to (3.4.9), we obtain

 + ρ = ∇*k***** + ρ **M** + ε*ikl* σ*lk*. (3.4.22)

By virtue of momentum conservation, the differential angular momentum equation is finally written in generalized form as

ρ = ∇*k***** + ρ **M** + **** (3.4.23)

(**** ≡ ε*ikl* σ*lk* ≡ (σ32 – σ23) + (σ13 – σ31) + (σ21 – σ12)).

It is clear that the stress tensor is generally not symmetric when intrinsic spin, body couples, or surface couples are present. The branch of continuum mechanics that studies the behavior of materials exhibiting these properties is called *mechanics of* *micropolar* (or *Cosserat*) *media* or *Cosserat continuum mechanics*.

In what follows, intrinsic spin, surface couples, and body couples are disregarded; i.e., the stress tensor is assumed to be symmetric:

σ*mk* = σ*km*. (3.4.24)

**§5. Integral and Differential Total Energy Equations**

The energy per unit mass of a continuous medium is the sum of internal energy *u* and kinetic energy *v*2 per unit mass:

*E* = *u* + *v*2. (3.5.1)

For a representative field *E*(*x*, *t*), the total energy ε*V* and rate of its change within an Eulerian control volume *V*E are expressed as

ε*V* = d*V*, (3.5.2)

 = d*V* = d*V*.

First, the energy ε*V* within *V*E can be changed by the inward or outward energy transfer due to mass transfer across its boundary *S*E. This mechanism is known as convective energy transfer. The rate of energy transfer, given by expression (1.4.15) with *f* = *E*, is

–*E* d*s* (*jn* = ρ*vn* = ρ*vknk*). (3.5.3)

Second, the energy ε*V* can be changed by work done on *V*E by external forces. In particular, work can be done by a contact force on *S*E at the rate given by the dot product of the force **σ***n* d*s* with linear velocity **v**,

d*s*, (3.5.4)

and by body forces **F** acting within *V*E, including gravitational, electromagnetic, and inertial ones (in a non-inertial frame of reference), with work rate (power) given by the dot product of the force ρ**F** with velocity **v** (cf. (3.3.3)),

d*V* (**F** = **g′** + **R**, **g′** = **g** + ****). (3.5.5)

Third, the energy within *V*E can be changed by the flow of heat across *S*E due to thermal conduction. As distinct from convective transfer and work done by external forces, heat conduction does not involve any bulk motion of matter. The rate of heat transfer across *S*E is expressed by the integral

–d*s*. (3.5.6)

Similarly to mass flow rate, heat flow rate is described here in terms of the heat flux vector **q**: the rate of heat transfer across an infinitesimal surface d*s* with normal **n** is

*qn* d*s* = *qk* *nk* d*s*. (3.5.7)

This expression, as well as the fact that  is the component of **q** across a surface element with normal **n**, follows from an analysis analogous to that performed to derive expression (2.2.15) for the traction  on a surface element with normal vector **n** (see Fig. 2.2.1).

Fourth, the energy within *V*E can be changed by effects of electromagnetic fields other than ponderomotive displacement of particles, whose contribution (ρ**R** **⋅** **v**) is subsumed in (3.5.5). These effects include dissipative processes, such as absorption of high-frequency electromagnetic waves (e.g., light) and Joule heating by electric currents flowing through the medium, which increase its thermal energy. Internal energy can also increase when electric or magnetic polarization is changed by an external electromagnetic field. Let  represent the conversion rate[[2]](#footnote-2) between electromagnetic energy and internal energy *u* per unit mass of the medium. Since the  field is supposed to be representative, the energy conversion rate within *V*E is expressed as

d*V*. (3.5.8)

Therefore, with added work done by the electromagnetic (ponderomotive) forces represented by an **R** field contributing to the body force field **F** (see (3.3.3)), the total rate of electromagnetic energy conversion per unit mass is expressed as the work rate density (power density)

 = **R** ⋅ **v** + . (3.5.9)

In summary, the total energy equation for an Eulerian control volume *V*E is written as

*E* d*V* =*E* *vknk* + **σ***k* *nk* **⋅** **v** – *qk* *nk*) d*s*

+ (**F** **⋅** **v** + ) d*V*. (3.5.10)

This energy balance equation, analogous to mass balance (3.2.2), momentum balance (3.3.5), and angular momentum balance (3.4.3), can be expressed in words as follows:

**Theorem (*energy conservation*)**. The energy within a time-independent domain can be changed by

**-** the energy transfer due to mass transfer across the domain boundary;

**-** the work done by external contact forces on the boundary;

**-** the work done by body (gravitational, electromagnetic, inertial) forces within the domain;

**-** conversion between electromagnetic and internal energy;

**-**heat transfer across the boundary.

The total energy balance for an arbitrary material volume *V*L is expressed by an equation analogous to (3.2.7), (3.3.5), and (3.4.4):

*E* d*V* = **⋅ v** – *qn*) d*s* + + (**F** **⋅** **v** +) d*V*. (3.5.11)

By applying the Gauss–Ostrogradsky theorem, the surface integrals are rewritten as volume integrals. Then, as in the derivations of the mass, momentum, and angular momentum equations, Theorem 3 and expressions (3.3.3), (3.5.5), and (3.5.9) are used to obtain the total energy equation

ρ*E* + ∇*k* (ρ*E* *vk*) = ∇*k* (σ*k* **⋅** **v**) – ∇*k* *qk* + ρ **g′** **⋅** **v** + ρ **R** **⋅** **v** + ρ. (3.5.12)

By substituting *f* = *E* into Eq. (3.2.10) and using definition of *E* in (3.5.1), the differential equation of total energy balance is represented as

ρ = ∇*k* (σ*kl* *vl*) – ∇*k* *qk* + ρ*vk* + ρ *Rk* *vk* + ρ ≡

≡ ρ+ ρ+ ρ + ρ, (3.5.13)

ρ ≡ ∇*k* (σ*kl* *vl*), ρ ≡ ρ **g′** ⋅ **v**,

ρ ≡ ρ + ρ **R** ⋅ **v**, ρ ≡ ρ≡ – ∇*k* *qk*,

where the work rates , , ,  and the heat transfer rate ≡  are defined per unit mass.

Thus, the total energy of a continuum particle can be changed by

* the work done at a rate  by external forces on the particle surface;
* the work done at a rate  by external gravitational and inertial forces;
* heat transfer across the particle surface at a , expressed in terms of the heat flux *qk*;
* electromagnetic energy conversion whose rate  combines the ponderomotive contribution ρ**R ⋅** **v** and an energy rate  associated with polarization, magnetization, charge displacements, and other phenomena unrelated to particle displacement.

**§6. Internal Energy Equation and the First Law of Thermodynamics**

Subtracting kinetic energy balance (3.3.11) from total energy balance (3.5.13) and noting that

∇*k* (σ*lk* *vl*) = σ*lk*∇*kvl* + *vl* ∇*k* σ*lk*, (3.6.1)

one obtains the following internal energy balance equation (sometimes referred to as heat flux equation) written per unit volume:

ρ = − ρ + ρ + ρ (3.6.2)

(ρ ≡ − σ*lk*∇*k* *vl*, ρ ≡ – ∇*kqk*).

Here,  is the rate of *work done by internal surface forces* per unit mass of the medium. Since this quantity is the contracted product of two tensors, σ*kl* and ∇*kvl*, it is invariant under rotations of the observer's coordinate system (see the outer product theorem in Appendix 1, **§**8). Furthermore, unlike , it is also invariant under translation (Galilean transformation **v**′ = **v** + **v**0 with constant **v**0) since the stress tensor σ*kl* is Galilean invariant and so is ∇*k* *vl*:

∇*k*  = ∇*k*(*vl* + *v*0*l*) = ∇*k* *vl*. (3.6.3)

In view of the symmetry of the stress tensor σ*kl*, the rate of work done by internal surface forces per unit mass can be represented as

− ρ = σ*lk* ∇*k* *vl* = σ*kl* ∇*l* *vk* = σ*kl*(∇*k* *vl* + ∇*l* *vk*) = σ*kl* *ekl*. (3.6.4)

Therefore, work is done by internal surface forces when the internal energy of a continuum particle is changed by deformation. When no deformation occurs (*ekl* = 0), the work of internal forces *A*(int) is zero.

Thus, Eq. (3.6.2) implies that a change in the internal energy of a continuum particle may occur as a result of work done by internal forces at a rate , heat transfer across the particle surface at a rate  , and conversion between electromagnetic and internal energy at a rate  unrelated to particle displacement. Similarly to , the power density  can be considered as the rate of work due to electromagnetic forces acting on the particle.

By virtue of identity (3.6.1), kinetic energy balance (3.3.11) in a region where velocity and stress are continuously differentiable can be equivalently written as follows:

ρ = ρ + ρ + ρ **F** ⋅ **v** ≡

≡ ρ( + ) + ρ( – ) + ρ (3.6.5)

≡ ∇*k*(σ*klvl*), ρ = − σ*kl* *ekl*,

 = + **R** ⋅ **v**,  = **g′** ⋅ .

Thus, a change in the kinetic energy of a continuum particle may occur as a result of work done by external body forces at a rate ρ**F** **⋅** **v** (which combines the work rate **g′** **⋅** **v** of gravitational plus inertial force and the work rate **R** **⋅** **v** of electromagnetic forces), as well as of work done by external and internal surface forces at rates  and , respectively.

Hereinafter, the following compact notation is used for the power densities that determine the rate of change of internal energy:

 ≡ ,  ≡ ,  ≡ . (3.6.6)

In thermodynamic terms, the total energy balance expressed by Eqs. (3.5.10)–(3.5.13), or the ensuing internal energy equation (3.6.2), are referred to as the *first law of thermodynamics*.

The principles of continuum thermodynamics are discussed in Chapter 5.

**§7. Integral and Differential Equations for Internal Energy and Entropy**

The mass, linear momentum, angular momentum, and total energy of a system can only be changed by external factors and fields. One of these is mass transfer across its boundary, which involves concomitant transfer of linear momentum, angular momentum, energy, and other extensive properties. Others include work done by external contact forces and heat transfer. Field contributions may be due to gravitational, inertial, and ponderomotive forces as well as to conversion between internal and electromagnetic energy. However, unlike mass, linear momentum, angular momentum, and total energy, some properties can be changed by internal factors. In particular, internal energy can be changed by work done by internal forces, while energy dissipation due to interaction between continuum particles can increase the entropy of the system in question (see Chapter 5).

The internal energy balance in the Eulerian control volume *V*E bounded by a surface *S*E is expressed by an integral equation analogous to those for mass, linear momentum, angular momentum, and total energy:

*u* d*V* = –*vn* *u* d*s* – d*s* + d*V* − d*V*. (3.7.1)

Here, the first term on the right-hand side represents the inflow of internal energy due to mass transfer across *S*E, the second term represents the heat inflow across *S*E, the third one is the rate of conversion of electromagnetic energy into internal energy, and the last term is the rate of work done by internal forces to change internal energy by deformation.

The work done by external contact forces changes the total energy at a rate . However, internal or kinetic energy may change even without any work being done by external forces ( = 0) when conversion between kinetic and internal energy occurs due to work done by internal forces. The expressions for  and  in Eq. (3.6.5) are valid when the stress and velocity fields are continuously differentiable.

Equation (3.7.1) can also be obtained by integrating the local internal energy balance in (3.6.2) over *V*E and performing an analysis analogous to that leading from (3.5.10) to (3.5.12) in reverse order. However, Eq. (3.7.1) derived in this manner is valid only if all functions therein are continuously differentiable on *V*E as required for the existence of derivatives in (3.6.2) and for Gauss–Ostrogradsky theorem (3.1.25) to be applied when changing from volume to surface integrals in the first two terms on the right-hand side of (3.7.1) (also recall time derivative theorem (3.1.5)).

The analysis of internal energy balance performed for an arbitrary control volume to derive Eq. (3.7.1) makes this equation valid when it contains functions discontinuous on *V*E, in particular when there exist jump surfaces in *V*E.

Let us perform an analogous analysis of the balance of entropy  for an Eulerian control volume *V*E. Like mass, momentum, total energy, and internal energy, entropy is an extensive quantity:

 = *s* d*V*, (3.7.2)

where *s* is specific entropy (i.e., entropy per unit mass of the medium).

A more general discussion of entropy in the context of the second law of thermodynamics is presented in Chapter 5, where physicochemical transformations and diffusion are taken into account. The present analysis is based on the simple notion of entropy as a thermodynamic state variable. In this simplified framework, the change in entropy is determined by two factors: (1) the amount of incoming or outgoing heat divided by the temperature at every location where heat is being added or removed, respectively, and (2) the rate of internal entropy production per unit mass, Φ ≥ 0. Entropy is produced internally at every point in *V*E where dissipative processes reduce a temperature gradient and convert mechanical or electromagnetic energy into heat. Accordingly, the entropy balance for *V*E is expressed by the integral equation

*s* d*V* = –*vn* *s* d*s* – d*s* + Φ d*V*, Φ ≥ 0. (3.7.3)

Here, the first term on the right-hand side represents the inflow of entropy due to mass transfer across *S*E, the second term represents the change in entropy due to heat inflow, and the third one is the rate of internal entropy production in *V*E with source strength Φ ≥ 0.

For a Lagrangian (material) control volume *V*L, the entropy balance is written analogously as follows:

*s* d*V* = –d*s* + Φd*V*, Φ ≥ 0. (3.7.4)

It important that internal entropy production due to interaction between continuum particles within a control volume makes entropy balance, as well as internal energy balance, essentially different from linear momentum, angular momentum, or total energy balance. The last four properties can only be changed by transfer across the boundary or by external (gravitational, inertial, electromagnetic) effects, not by interactions within the region.

From the last integral equation, the following local (differential) balance equation can be derived for specific entropy:

ρ = – ∇*k*  + ρ Φ (Φ ≥ 0). (3.7.5)

**§8. A General Form of Integral and Differential Balance Equations**

For the Eulerian (stationary) control volume *V*E bounded by a surface *S*E, the balances of mass (3.2.2), linear momentum (3.3.4), angular momentum (3.4.20), total energy (3.5.10), and entropy (3.7.3) are written as follows:

d*V* = −d*s*, (3.8.1)

**v** d*V* =d*s* + **F** d*V*, (3.8.2)

d*V* = (3.8.3)

=d*s* + ([**x** × **F**] + **M**) d*V*,

d*V* = (3.8.4)

=*nk* d*s* + d*V*,

*s* d*V* = *nk* d*s* + Φ d*V*. (3.8.5)

Their Lagrangian counterparts written for the control volume *V*L bounded by a (possibly moving) surface *S*L consisting of material points have the form

d*V* = 0, (3.8.6)

**v** d*V* = d*s* + **F** d*V*, (3.8.7)

([**x** × **v**] + **m**) d*V* = [**x** × **σ***k*] + **θ***k*) *nk* d*s* + (3.8.8)

+([**x** × **F**] + **M**) d*V*,

 d*V* = **σ***k* ⋅ **v** – *qk*) *nk* d*s* + (**F** ⋅ **v** + ) d*V*, (3.8.9)

*s* d*V* = )*nk* d*s* + Φ d*V* (Φ ≥ 0). (3.8.10)

Note that the source terms in the balance equations for conserved properties distributed over *V*E or *V*L (mass, linear momentum, angular momentum, and total energy) represent the effects of external (gravitational, inertial, or electromagnetic) fields, whereas the source term in the entropy balance represents its internal production due to interaction between continuum particles within *V*E or *V*L.

The integral equations can be used derive differential balance equations for mass (3.2.6), linear momentum (3.3.9), angular momentum (3.4.23), total energy (3.5.13), and entropy (3.7.5), which are valid locally in the domain of motion where the time and spatial derivatives contained in the equations are defined:

 + ∇*k* ρ *vk* = 0, (3.8.11)

ρ = ∇*k*  + ρ **F**, (3.8.12)

ρ = ∇*k* **θ***k* + ρ**M** +[**σ**] ([**σ**] = **e***i* ε*ikm* σ*mk*), (3.8.13)

ρ =  + ρ(**F** ⋅ **v** + ), (3.8.14)

ρ = – ∇*k*  + ρ Φ (Φ ≥ 0), (3.8.15)

( ≡  + *vk*∇*k* или ρ ≡  + ∇*k* ρ *f* *vk*). (3.8.16)

The respective source terms ρ**F**, ρ**M**, and ρ(**F** ⋅ **v** + ) arising in the linear momentum, angular momentum, and total energy equations represent the influence of *extraneous* bodies mediated by gravitational or electromagnetic fields. In contrast, the source term ρΦ in the entropy balance equation represents both effects of extraneous bodies and those of *internal* processes in the medium (see Chapter 5, §§14 and 15).

Let us introduce generalized (per unit mass) quantities: *f* is mass, linear momentum, angular momentum, total energy, or entropy; ψ*k* represents surface forces and fluxes;  represents the action of gravitational, inertial, and electromagnetic fields; and  represents internal production, which is zero for mass, linear momentum, angular momentum, and total energy but may contribute to the balance of kinetic energy ( = − σ*kl* *ekl*), internal energy ( = σ*kl* *ekl*), or entropy ( = Φ ≥ 0). Expressions for *f*, ψ*k*, , and  are listed in the table below, where each column corresponds to a particular balance equation (for mass, linear momentum, angular momentum, total energy, kinetic energy, internal energy, or entropy).

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| *f* | 1 | **v** | [**x** × **v**] + **m** | *u* + |  | *u* | *s* |
| ψ*k* | 0 |  |  | – *qk* |  | – *qk* | – |
| ρ | 0 | ρ | ρ+ ρ**М** | ρ + ρ |  | ρ | 0 |
| ρ | 0 | 0 | 0 | 0 | − |  | Φ ≥ 0 |

(3.8.17)

For an Eulerian control volume *V*E, balance equations (3.8.1)–(3.8.5) are written in a unified form as

*f* d*V* = – ρ *vk* *f* + ψ*k*) *nk* d*s* +  d*V* + d*V*. (3.8.18)

Substituting *f*, ψ*k*, , and  from (3.8.17) column by column, one obtains the corresponding balance equations (3.8.1)–(3.8.5), respectively.

Analogous balance equations written for a Lagrangian control volume have the form

*f* d*V* = *nk* d*s* +  d*V* +  d*V*. (3.8.19)

Differential balance equations (3.8.11)–(3.8.17) can be written as

 + ∇*k*ρ*vk* = 0, ρ = ∇*k* ψ*k* + ρ +  (3.8.20)

.

1. To simplify notation, the traction **σn** on a surface element with normal vector **n** is hereinafter denoted by **σ***n*, where subscript *n* corresponds to the unit vector **n**, as distinct from subscripts *i*, *j* = 1, 2, 3. [↑](#footnote-ref-1)
2. The overdot is frequently used to denote a time derivative. With an overdot, characters representing work, heat, and other forms of energy stand for the corresponding temporal rates of change. [↑](#footnote-ref-2)